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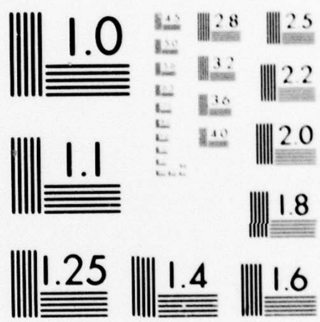

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1. Introduction

A voting game $G = (N,W)$ is defined here as a strong and proper simple game ~~as given in Shapley [4,5].~~ N is a finite set of voters (players) and W is a collection of subset of N called winning coalition such that:

i) $w \in W \Rightarrow s \in W \quad \forall\, w \subset s \subset N$

ii) $w \in W \Rightarrow \bar{w} \notin W$

where $\bar{w} = N - w$.

A winning coalition w is a subset of N which can guarantee a victory on any issue if all the members in w cooperate. A coalition which is not winning is called a losing coalition.

Consider two parties which are competing for votes in a voting game $G = (N,W)$. Let A be the party which wishes to buy "pro" votes, and B the one which wants to buy "con" votes. A voter $i \in N$ will side with the party which pays him more. I.e.,

if A chooses the allocation vector $P = (p_1, p_2, \ldots, p_n)$ and

if B chooses the allocation vector $Q = (q_1, q_2, \ldots, q_n)$ where

p_i is the amount paid by A to voter i and q_i is the amount paid by B to voter i; then i will side with A if $p_i > q_i$ while i will vote for B if $p_i < q_i$. There will be a tie with probability 0.5 of voter i going either way when $p_i = q_i$.

The winning party is the one which has a winning coalition in its favor. The problem is to find an optimal strategy for each party when each one has the same total financial resources.

For any allocation vectors P and Q we have

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1 \quad \text{and} \quad p_i \geq 0, \; q_i \geq 0 \quad \text{for all } i.$$

Let S be the space of all allocation vectors

$$S = \{X \in R^n_+ | \sum_{i=1}^{n} x_i = 1\}$$

where R^n_+ is the set of nonnegative n-dimensional vectors with real components. The symmetry of the game implies that if there is an optimal strategy for one party it is also an optimal strategy for the other party.

Let us assume here that an optimal strategy exists and can be represented as the Lebesgue integral of some continuous function f which is positive inside some connected subset M of S and zero outside it.

For any P and $Q \in M$ define

$$w(P,Q) = \{i | p_i > q_i\}.$$

Then party A wins if $w(P,Q) \in W$ while party B wins if $w(Q,P) \in W$.

Some Basic Lemmas

Lemma 1: A necessary condition for an optimal strategy to exist and for it to be represented by function f over the subset M of S is that for any $Q \in M$ we have

$$\int_{S_i} f \, dv_i = \int_{S_j} f \, dv_j \qquad 1 \leq i < j \leq N$$

where

$S_k = \{X \in M | x_k = q_k \text{ and } w(X,Q) \cup \{k\} \in W^*\}$

W^* is the set of minimal winning coalitions, and

v_k is the Lebesgue measure at $v_k = q_k$.

Proof: We have

Prob(Party A wins | Party A chooses an optimal strategy and Party B chooses a strategy $Q \in M$) = 1/2

i.e.

$$\int_{S_A^Q} f \, dv = 1/2 \tag{1}$$

where

$$S_A^Q = \{X \in M | w(X,Q) \in W\}.$$

Now if Q is an interior point in M and party B changes its strategy to $Q' \in M$ where

$$Q' = Q + \delta Q$$

then we get as before

$$\int_{S_A^{Q'}} f \, dv = 1/2 \tag{2}$$

By subtracting (1) from (2), for small $||\delta Q||$, gives

$$\sum_{i=1}^{n} \delta q_i \int_{D_i} f \, dv_i \approx 0$$

where

$$\begin{aligned} D_i &= \{X \in M | x_i = q_i, \; w(X,Q) \cup \{i\} \in W\} - \{X \in M | x_i = q_i, \; w(X,Q) \in W\} \\ &= \{X \in M | x_i = q_i, \; W(X,Q) \cup \{i\} \in W^*\} \\ &= S_i. \end{aligned}$$

Then

$$\sum_{i=1}^{n} \delta q_i \int_{S_i} f\, dv_i \not\geq 0 \qquad (3)$$

Since the δq_i's are arbitrary except that $\sum_{i=1}^{n} \delta q_i = 0$, this implies that

$$\int_{S_i} f\, dv_i = \int_{S_j} f\, dv_j \qquad 1 \leq i < j \leq N.$$

<u>Lemma 2</u>: If an optimal strategy f exists then the closure of M, denoted by $\overline{M}$, has a non-empty intersection with each simplex S_w where

$$S_w = \{X \in S \mid \sum_{i \in w} x = 1, \quad w \in W^*\}$$

This lemma can be restated as follows.

For any minimal coalition $w \in W^*$ there is a strategy $Q \in \overline{M}$ such that

$$q_i = 0 \text{ for all } i \notin w.$$

<u>Proof</u>: Assume $\overline{M}$ has an empty intersection with a simplex S_w for some $w \in W^*$. Choose $Q \in \overline{M}$ such that

$$\sum_{i \in w} q_i = \sup_{x \in M} \sum_{i \in w} x_i .$$

Then

$$\epsilon \equiv 1 - \sum_{i \in w} q_i > 0.$$

Pick Q' so that

$$q_i' = q_i + \epsilon/|w| \qquad i \in w, \text{ and}$$

$$q_i' = 0 \qquad i \notin w.$$

Then

Prob(Party B wins coalition w | Party A chooses an optimal strategy and Party B chooses a strategy Q) = 1/2

< Prob(Party B wins coalition w | Party A chooses an optimal strategy and Party B chooses the strategy Q')

i.e.

Prob(Party B wins | Party A plays an optimal strategy) > 1/2

which is a contradiction of the definition of optimal strategy.

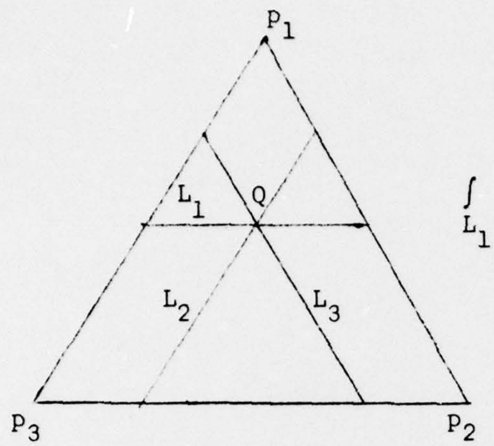
Example: Consider a symmetric simple game $G = (N,W)$ in which

$$w \in W \iff |w| > |N|/2$$

where $|w|$ is the number of players in coalition w, and $n = |N|$ is odd.

i) Case $n = 3$:

Each S_i is a line L_i through Q parallel to the side of the simplex S. By using lemmas (1) and (2), it is easy to show that the distribution of each component of the allocation vector P is uniform on the interval $(0,2/3)$.



$$\int_{L_1} f \, dv_1 = \int_{L_2} f \, dv_2 = \int_{L_3} f \, dv_3$$

ii) Case $n = 5$:

The distribution of each of P in this case is not uniform, since if the components of P were uniform one could choose Q as $(1/3,1/3,1/3,0,0)$. Then the probability that party A wins a voter i, $i = 1,2,3$, is $1/6$. Therefore the probability that party A wins at least one of the voters 1, 2, or 3 is less than $1/2$. So the probability that party A wins is less than $1/2$.

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